数理解析研究所講究録
255巻 1975年 33-41

Analogue of inverse scattering theory for the discrete Hill's equation and exact solutions for the periodic Toda lattice*)

Etsuro Date and Shunichi Tanaka
Department of Mathematics, Osaka University

Recently Dubrovin[1),2)] and Its-Matveev[3)] developed an analogue of the inverse scattering theory for the Hill's equation and have given explicit form of the periodic potentials with the finite number of gaps in the spectrum. The KdV equation is exactly solved in that class of potentials leading to the effective construction of the periodic N-solitons[2),3)].

In this note we describe briefly corresponging results for discrete Hill's equation and exact solutions of the periodic Toda lattice. Details will appear elsewhere.

1. Spectral properties of the discrete Hill's equation

Consider the discrete Hill's equation

$$L\ u = \lambda u,$$

$$(Lu)(n) = a_n u(n+1) + b_n u(n) + a_{n-1} u(n-1), \tag{1.1}$$

$$a_n>0;\ a_{n+N}=a_n,\ b_{n+N}=b_n.$$

---

*) At the time of the symposium, the second named author has given an exposition of Refs. 1),2),3). Instead of reproducing the contents of the talk, we give here discrete analogue of these results.

We define a fundamental system of solutions of (1.1), $y(n)$, $z(n)$, by the initial conditions

$$
\begin{aligned}
y(0) &= 1, & y(1) &= a_0^{-1}(\lambda-b_0), \\
z(0) &= 0, & z(1) &= a_0^{-1}.
\end{aligned}
\tag{1.2}
$$

Then for $n\geq 0$ $y(n)$ is a polynomial in λ of the form

$$y(n) = (\Pi_{j=0}^{n-1} a_j)^{-1}\{\lambda^n - (\Sigma_{j=0}^{n-1} b_j)\lambda^{n-1}+\cdots\} \tag{1.3}$$

and $z(n)$ is a polynomial in λ of the form

$$z(n) = (\Pi_{j=0}^{n-1} a_j)^{-1}\{\lambda^{n-1} - (\Sigma_{j=1}^{n-1} b_j)\lambda^{n-2}+\cdots\}. \tag{1.4}$$

For $n<0$ similar expressions hold.

By the periodicity of a_n, b_n, we have

$$
\begin{aligned}
\begin{bmatrix} y(n+N) \\ z(n+N) \end{bmatrix} &= M(\lambda) \begin{bmatrix} y(n) \\ z(n) \end{bmatrix} \\
M(\lambda) &= \begin{bmatrix} y(N) & -a_{N-1}y(N-1) \\ z(N) & -a_{N-1}z(N-1) \end{bmatrix}.
\end{aligned}
\tag{1.5}
$$

We call $M(\lambda)$ the monodromy matrix of the system (1.1), (1.2). Noting that det M is an analogue of Wronskian, we have det $M=1$.

Denote by $y(k,n)$, $z(k,n)$ the solutions of (1.1), (1.2) in which the coefficients a_n, b_n are replaced by a_{n+k}, b_{n+k}. Then the relations

$$
\begin{aligned}
y(k,n) &= a_{k-1}y(k-1)z(k+n) - a_{k-1}z(k-1)y(k+n), \\
z(k,n) &= y(k)z(k+n) - z(k)y(k+n)
\end{aligned}
\tag{1.6}
$$

hold.

Using these relations we have

$$y(N)-a_{N-1}z(N-1) = y(k,N)-a_{k-1}z(k,N-1), \tag{1.7}$$

namely the trace of the monodromy matrix of the translated system is equal to the original one. Denote (1.7) by $\Delta(\lambda)$.

The roots of the equation

$$\Delta(\lambda)^2 - 4 = 0 \tag{1.8}$$

are all real and are ordered as

$$\lambda_1<\lambda_2\le\lambda_3<\cdots<\lambda_{2j}\le\lambda_{2j+1}<\cdots<\lambda_{2N-2}\le\lambda_{2N-1}<\lambda_{2N}.$$

The roots of the equation

$$z(k,N) = 0$$

are real and distinct. If we order them by

$$\mu_1(k)<\mu_2(k)<\cdots<\mu_{N-1}(k),$$

we have $\mu_j(k) \in [\lambda_{2j}, \lambda_{2j+1}]$.

Comparing the coefficients of λ^{N-2} of $z(k,N)$, we have

$$\Sigma_{j=0}^{N-1} b_j - b_k = \Sigma_{j=1}^{N-1} \mu_j(k). \tag{1.9}$$

Introducing the notations

$$\Lambda = 2^{-1} \Sigma_{j=1}^{2N} \lambda_j,$$

and comparing the coefficients of λ^{2N-1} of $\Delta(\lambda)^2-4$, we have

$$\Sigma_{j=0}^{N-1} b_j = \Lambda. \tag{1.10}$$

We define Bloch eigenfunction of (1.1) by

$$x_\pm(n) = y(n) + \frac{-\{a_{N-1}z(N-1)+y(N)\}\pm(\Delta(\lambda)^2-4)^{1/2}}{2z(N)} z(n).$$

By the direct calculation using (1.5), (1.6), we have

$$x_+(k)x_-(k) = z(k,N)/z(N). \tag{1.11}$$

2. Hyperelliptic abelian integrals and the solution of Jacobi's inversion problem

In what follows among roots of (1.8) simple roots play important roles. Assuming their number to be 2g+2 and changing the numbering, we denote simple roots by

$$\lambda_1<\lambda_2<\cdots<\lambda_{2g+2}$$

and double roots by $\lambda_{2j+1}=\lambda_{2j+2}$ $(j=g+1,\cdots,N-1)$. We also change the numbering for $\mu_j(k)$ so that the relations

$$\lambda_{2j}\le\mu_j(k)\le\lambda_{2j+1}, \qquad j=1,\cdots,g,$$

$$\mu_j(k) = \lambda_{2j+1} = \lambda_{2j+2}, \qquad j=g+1,\cdots,N-1$$

hold.

Consider the Riemann surface of the hyperelliptic curve

$$\mu^2 = R(\lambda) = \Pi_{j=1}^{2g+2} (\lambda-\lambda_j).$$

This Riemann surface is realized by cross-connecting two copies of the complex λ-planes which are cut along the intervals $(\lambda_{2j-1}, \lambda_{2j})$, $j=1,\cdots,g+1$. We mean by the upper sheet the sheet on which $R(\lambda)^{1/2}$ is positive on (λ_{2g+2},∞). For the point $\lambda\in S$ the corresponding point on the other sheet is denoted by λ'.

On this surface S we take a system of canonical cuts α_j, β_j, $j=1,\cdots,g$.

For α_j we take a closed contour which starts at λ_2, goes on the upper sheet as far as λ_{2j+1}, crosses to the lower sheet and ends at λ_2. For β_j we take a closed contour which surrounds the cut $(\lambda_{2j+1}, \lambda_{2j+2})$, $j=1,\cdots,g$ on the upper sheet. Bloch eigenfunctions $x_+(k)$ and $x_-(k)$ can be regarded as the branch of the single-valued function $x(k)$ on S. By (1.11) and the asymptotic properties of $x_\pm(k)$ as $\lambda\to\infty,\infty'$, we have

Theorem. $x(k)$ has g simple zeros at $\mu_j(k)$ $(j=1,\cdots,g)$, each of which denotes a point on S whose projection on the complex plane is $\mu_j(k)$ (so $\mu_j(k)'$ is not zero of $x(k)$). It has g simple poles at $\mu_j(0)$, has zero of k-th order at ∞' (on the lower sheet) and has pole of k-th order at ∞ (on the upper sheet).

Define a base of the abelian differentials of the first kind

$$\omega_m = \Sigma_{j=0}^{g-1} c_{mj}\lambda^j R(\lambda)^{-1/2}d\lambda, \qquad m=1,\cdots,g,$$

normalized by

$$\int_{\beta_j} \omega_m = -\pi i\delta_{jm}, \qquad j,m=1,\cdots,g.$$

Put

$$\int_{\alpha_j} \omega_m = t_{jm}, \qquad j,m=1,\cdots,g.$$

Then $c_{m,j}$ are real and the matrix (t_{jm}) is a real symmetric negative definite matrix.

Let Γ be the lattice in C^n spanned by vectors:

$$\left.\begin{array}{l}(t_{j1},\cdots,t_{jg}),\\ (0,\cdots,\underset{j}{\pi i},\cdots,0),\end{array}\right. \qquad j=1,\cdots,g.$$

The complex torus $J=C^n/\Gamma$ is called the Jacobian variety of the hyperelliptic curve $\mu^2=R(\lambda)$.

Making use of Theorem (see Akhiezer[4]), we have

$$\Sigma_{j=1}^{g} \int_{\mu_0}^{\mu_j(k)} \omega_\ell \equiv k\int_{\infty}^{\infty'}\omega_\ell + \Sigma_{j=1}^{g} \int_{\mu_0}^{\mu_j(0)} \omega_\ell \bmod \Gamma \tag{2.1}$$

where μ_0 is a fixed point on S.

We introduce Riemann theta function defined by

$$\theta(u) = \Sigma_{m_1,\cdots,m_g=-\infty}^{\infty} \exp[2\Sigma_{j=1}^{g} m_j u_j + \Sigma_{j,k=1}^{g} t_{jk}m_j m_k],$$

$$u = (u_1,\cdots,u_g) \in C^g.$$

Solution of Jacobi's inversion problem permits us to express symmetric polynomials of $\mu_j(k)$ by the right hand side of (2.1) as rational function of theta functions. Following Its-Matveev[3] we write first of them as

$$\begin{aligned}\Sigma_{j=1}^{g} \mu_j(k) = \pi^{-1}i\Sigma_{j=1}^{g} \int_{\beta_j} \lambda\omega_j &+ \\ &+ \Sigma_{j=1}^{g} c_{j,g-1}D_j \log\theta(u(\infty)+kc+d) \\ &- \Sigma_{j=1}^{g} c_{j,g-1}D_j \log\theta(u(\infty)+(k+1)c+d)\end{aligned} \tag{2.2}$$

where

$$u(\lambda) = (\int_{\mu_0}^{\lambda}\omega_1,\cdots,\int_{\mu_0}^{\lambda}\omega_g),$$

$$c = (-\int_{\infty}^{\infty}, \omega_1, \cdots, -\int_{\infty}^{\infty}, \omega_g),$$

$$d = (d_1, \cdots, d_g),$$

$$d_j = -\Sigma_{k=1}^{g} \int_{\mu_0}^{\mu_k(0)} \omega_j + 2^{-1} j\pi i - 2^{-1}\Sigma_{k=1}^{g} t_{kj},$$

and D_j denotes the partial differentiation with respect to the j-th variable.

By (1.9), (1.10), we have

$$b_k = \Lambda^* - \Sigma_{j=1}^{g} \mu_j(k) \tag{2.3}$$

$$\Lambda^* = \Sigma_{j=1}^{2g+2} \lambda_j.$$

(2.2) and (2.3) give explicit formula for the coefficients b_k in terms of the Riemann theta functions.

3. Integration of the periodic Toda lattice

The equation of motion of Toda lattice[5] has the form

$$\dot{Q}_n = P_n, \qquad \dot{P}_n = -\{\exp(-(Q_{n+1}-Q_n)) - \exp(-(Q_n-Q_{n-1}))\},$$

where the dot denotes the differentiation with respect to the time variable t.

Putting

$$a_n = 2^{-1}\exp\{-(Q_n-Q_{n-1})/2\}, \qquad b_n = -2^{-1}P_{n-1},$$

these equations take the following forms

$$\dot{a}_n = a_n(b_{n+1}-b_n), \qquad \dot{b}_n = 2(a_n^2-a_{n-1}^2).$$

These equations are equivalent to the evolution equation of linear operators[6)]

$$\dot{L} = [\ B,\ L\ ] = BL - LB$$

where

$$(Bu)(n) = a_n u(n+1) - a_{n-1} u(n-1).$$

Using these expressions we have

$$\begin{aligned} \dot{y}(n) &= a_n y(n+1) + (b_0 - \lambda) y(n) - a_{n-1} y(n-1) + 2a_{N-1}^2 z(n), \\ \dot{z}(n) &= -2y(n) + a_n z(n+1) + (\lambda - b_0) z(n) - a_{n-1} z(n-1). \end{aligned} \tag{3.1}$$

From these formulas, we have

$$\dot{\Delta}(\lambda) = 0$$

i.e. λ_j are independent of t.

So Riemann surface and the normalized differentials on it introduced in §2 are also independent of t (namely they are determined by the initial conditions). In the construction of §2 dependency on t comes only through $\mu_j(0)$.

By (3.1), we have differential equation for $\mu_j(0)$:

$$\dot{\mu}_j(0) = -2R(\mu_j(0))^{1/2} \Pi_{k=1,\neq j}^{g} (\mu_j(0) - \mu_k(0))^{-1} \tag{3.3}$$

Differentiating $d_j(t)$ with respect to t, inserting (3.3) and then using the Lagrange's interpolation formula, we have $\dot{d}_j = 2c_{n,g-1}$ i.e.

$$d_j(t) = d_j(0) + 2c_{n,g-1} t.$$

By (2.2), (2.3), we have

$$b_n(t) = \Lambda^* - \pi^{-1} i \Sigma_{j=1}^{g} \int_{\beta_j} \lambda\omega_j - 2^{-1} d/dt \log \frac{\theta(u(\infty)+nc+d(t))}{\theta(u(\infty)+(n+1)c+d(t))} .$$

Formulas for $a_n(t)$, $P_n(t)$ and $Q_n(t)$ are direct consequences of this formula.

References

1) B.A. Dubrovin, Funct. Anal. Appl. 9 (1975), 65.

2) B.A. Dubrovin and S.P. Novikov, J. Exp. Theor. Phys. 67 (1974), 2131.

3) A.R. Its and V.B. Matveev, Funct. Anal. Appl. 9 (1975), 69; Theor. Math. Phys. 23 (1975), 51.

4) N.I. Akhiezer, Sov. Math. Doklady 2 (1961), 1409.

5) M. Toda, Studies of a non-linear lattice, Physics Reports, 18C (1975).

6) H. Flaschka, Prog. Theor. Phys. 51 (1974), 703.